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The pinwheel pattern and its application to the manufacturers' pallet-loading problem

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Abstract

To solve the manufacturer's pallet-loading problem (MPLP), pallet-loading patterns with regular, sound and optimum number layouts should be presented. In this paper, we present a pinwheel pattern as an alternative solution to the pallet-loading problem. The definition, elements, categories, and practical advantages, generating algorithms of the pinwheel patterns, are discussed, and a uniform notation is proposed. With the ranges for all pinwheel instances within an area ratio no more than 76 boxes calculated, the study of geometry shows that each pinwheel pattern has a specific range of box ratio, and it may achieve optimality. The pinwheel pattern can be found for all non-prime numbers of boxes. Further discussions are focused on the dataset, loophole constraint and asymmetric pinwheels. The study suggests that the pinwheel pattern is an advantageous alternative to implement the MPLP.

Keywords: packing; pallet-loading pattern; geometry; pinwheel

1. Introduction

The manufacturer's pallet-loading problem (MPLP; e.g. Dowsland, 1987a; Morabito and Morales, 1998) is a well-known type of a two-dimensional cutting and packing problem, in which a single pallet has to be loaded with a maximal number of identical boxes. The problem has many practical applications in production, distribution and logistics. Dyckhoff (1990) defined this problem as of type 2/B/O/C, which means the problem is to assign a maximal number of small identical rectangles to a given large rectangle (representing the pallet). Wächer et al. (2007) further define the MPLP as a two-dimensional, rectangular identical item packing problem, and noted it as a layout problem regarding the arrangement of the (identical) small items on each of the large objects with respect to the geometric condition.

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MPLP problems can be formulated as special 0–1 LP models, such as Beasley's (1985). However, because of the size of practical instances, 0–1 models are generally too large to be computationally treated (Morabito et al., 2000). More , recently MPLP is just called the pallet-loading problem (PLP). In the 30-year process of pursuing a PLP solution, although there are exact algorithms (Dowsland, 1987a; Bhattacharya et al., 1998; Alvarez-Valdes et al., 2005; Martins and Dell, 2008), heuristics emerge more and often. To solve this problem, the optimal number of boxes along with their loading pattern, as verification, should be figured out.

The pallet-loading pattern is an arrangement of boxes oriented orthogonally on the pallet to achieve a regular, stable and optimum number layout. Many heuristics are concentrated on the loading pattern to find the optimal solution, from the early four-block algorithms (Steudel, 1979; Smith and de Cani, 1980) to G4 (Scheithauer and Terno, 1996), G5 (Martins and Dell, 2008), M&M heuristic (Morabito and Morales, 1998) and the L approach by Lins et al. (2003). Besides, there are metaheuristics based on tabu search (Pureza and Morabito, 2006), genetic algorithms (Herbert and Dowsland, 1996) and strategic oscillation (Amaral and Wright, 2001). Several upper bounds have also been proposed (Dowsland, 1984, 1985; Nelißen, 1995; Letchford and Amaral, 2001; Morabito and Farago, 1998), which consider the geometric structure of the problem and the linearly relax integer programming formulations.

Recently, Martins and Dell (2008) have solved optimality for all instances of PLP with an area ratio <101 boxes. Of the entire 3,080,730 equivalent classes of PLP instances, 86.2% of the classes solved are within the one-block (58.9% alone) or the two-block patterns. Although one-block and two-block patterns are common, they are usually of minor interest for pallet loading because of weak-stability guillotine cuts (Dyckhoff, 1990) and lack of symmetry for two-block patterns. Nevertheless, these classes and instances are based on the constraints proposed by Dowsland (1984) with the pallet length to width ratio between 1 and 2, the box length to width ratio between 1 and 4 and the pallet to box area ratio from 1 to 51 named as datasets Cover I, and from 51 to 101 as Cover II (Nelißen, 1995). However, in practical works, pallet ratios are finite and predetermined, while the box ratio may exceed 4. The more the area ratio, the more the possibilities of one-block optimal patterns, which tend to be unstable and of less practical value. Thus, it is time to re-examine the range of the box ratio and area ratio constraints for the PLP.

Although there are many different types of loading patterns (Arenales and Morabito, 1995), the loading pattern alone seldom draws enough attention from the researchers, whose main concern is algorithms for area utilization optimality. Because the optimality of all PLP instances within an area ratio <101 boxes has finally been solved by the integration of many algorithms (Martins and Dell, 2008), suitable alternatives must be selected and presented for implementation on the request of the Operations Research methodology (Winston, 1994). PLP algorithms usually lead to a pinwheel-loading pattern, which is a sound, textured and easy-to-implement pattern with an even longer history than PLP (Bolz and Hagernan, 1958); although it may not be optimal in the number of boxes loaded, it is a suitable alternative for implementation. Therefore, we conducted this alternative study on pinwheel patterns, hoping to find their characteristics and applications to PLP.

The structure of this paper is as follows: the next section describes the pallet-loading pattern, its definition, characteristics and advantages. Next, a generic notation with six parameters for the pinwheel patterns is proposed and the notation's applications to the three pinwheel categories, namely simple, block and nested pinwheels are identified. In the geometry study, the relationships between hole constraints, contour shape and box ratio are derived. With this geometry, the dataset and computation of all pinwheel instances within an area ratio of no more than 76 boxes are presented and discussed. Furthermore, asymmetric pinwheels are briefly examined. Finally, they are summarized and some issues are proposed for future study.

2. PLP and the pinwheel pattern

For PLP problems, any (X, Y, a, b) instance is defined, where X and Y are the length and width of the pallet, respectively, and a and b are the length and width of the box. It is assumed that at least one box can be placed in the pallet ($X \geqslant Y \geqslant a \geqslant b$). Without loss of generality, it is assumed that X, Y, a and b are positive integers (Bischoff and Dowsland, 1982). Here, for convenience, we define three ratios, which are just single Greek letters and different from Dowsland's (1987b):

Pallet ratio:

$$\alpha = X/Y. \tag{1}$$

Box ratio:

$$\beta = a/b. \tag{2}$$

Pallet to box area ratio:

$$\gamma = \left\lfloor \frac{XY}{ab} \right\rfloor, \tag{3}$$

where $\lfloor r \rfloor$ gives the largest integer that does not exceed r. α, β and γ are all ≥ 1. γ is also a simple but powerful upper bound, and will be used as a measure for the size of the problem Q4 (Bhattacharya et al., 1998).

2.1. PLP loading pattern

In PLP, boxes loaded on pallets should be orthogonal and without overlapping; accordingly, it is defined that the box is an H-box if it lies horizontally with its length a parallel to the length X of the pallet (for convenience, X is also the horizontal axis), its width b is parallel to the width Y of the pallet, and a vertically orientated box is called a V-box. A (homogeneous) block is defined as a rectangular subset of boxes that have the same orientation (V-boxes or H-boxes) (Scheithauer and Terno, 1996). Thus, the pallet-loading pattern is an arrangement of H-boxes/H-blocks and/or V-boxes/V-blocks on the pallet.

The loading pattern of PLP can be classified as guillotine, first-order non-guillotine and superior-order non-guillotine (Arenales and Morabito, 1995). A guillotine cut is a cut from one edge of a previously cut rectangle to the opposite edge (Dowsland and Dowsland, 1992); guillotine cuts apply to a rectangle orthogonally and produce two new rectangles. A pattern that results from successive guillotine cuts is a guillotine pattern. A guillotine pattern consists of one or more blocks; thus, there are one-block, two-block and three-block guillotine patterns. On the other hand, a first-order non-guillotine pattern has more than two homogeneous blocks, and a cut is of first order if it produces five new rectangles arranged in such a way as to not form a guillotine-cutting pattern (Arenales and Morabito, 1995). Early four-block heuristics and Morabito and Morales (1998) heuristic can generate optimal first-order non-guillotine patterns. There is also a superior-order non-guillotine pattern, which is a pattern that cannot be obtained by successive guillotine cuts and/or first-order non-guillotine cuts. Many complex algorithms are based on the superior-order non-guillotine pattern, such as the G and L algorithms mentioned above.

2.2. *Pinwheel-loading pattern*

The pinwheel pattern is a common and basic layout of H-boxes' and V-boxes' combined configuration for PLP. The four-block algorithm and many other algorithms mentioned above can produce pinwheel patterns. In a pinwheel pattern, there is a rectangular or a square contour and one or more inner holes. The contour is a rigid and straight rectangle enclosed by the outward edges of the outer boxes. A hole is an empty space left by the encompassed H-boxes and V-boxes in a loop. These H-boxes and V-boxes are called H-leaves and V-leaves, and there are always four leaves in a loop/hole unit. Figure 1 shows examples of a pinwheel pattern, which will be further discussed. For the first four-box simplest pinwheel (Fig. 1a), if its outer corners are folded inward, then it looks much more like the real pinwheel from which its name comes(Fig. 1b). Based on the leaves heights, there are two styles of patterns: P1 and P3, which conform to Steudel's heuristic (1979). If 1H is lower than 2V in height, then it is a P1 style (Fig. 1c), otherwise a P3 style (Fig. 1d).

In Fig. 1, the most complicated pinwheel pattern is 1H, which has six loops and six holes in two rows and three columns. In each loop, its H-leaf is an H-box, and the V-leaf a block of two V-boxes. Each of the two adjacent loops is shared by a common leaf. Furthermore, two single H-leaves are added in the northeast and southwest corners, and three single V-leaves forming an L shape are added in the southeast and northwest corners; altogether, these loops and additional

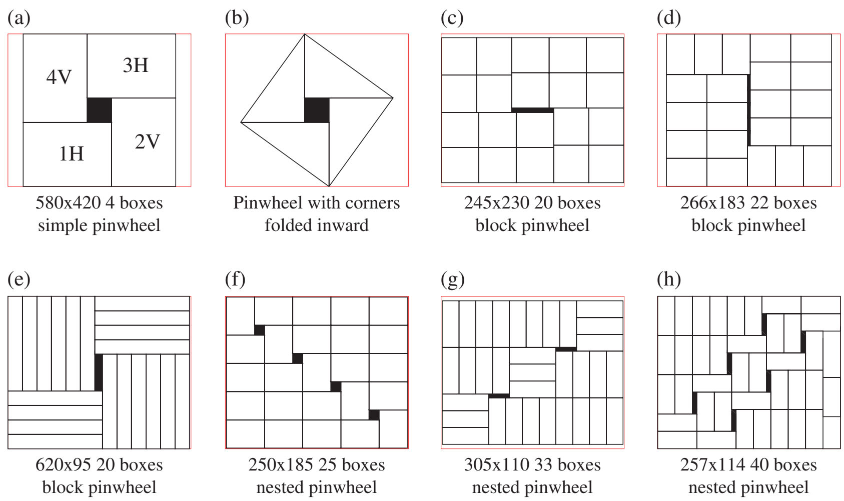

Fig. 1. Some examples of pinwheel patterns; all outer red rectangles represent 1200 × 1000 pallets.

H-leaves and V-leaves form the pattern's complete rectangle. Thus, the pinwheel pattern is characterized by the holes (loops) arrangement, leaves formation and orientation.

2.3. Practical application of pinwheel-loading pattern

The PLP is simplified as a two-dimensional layout problem for maximizing the area utilization. However, after the optimality is reached, the implementation of the pallet layout should revert to three dimensions again to consider the stacking layers and stability of the pallet load. This objective conversely has other requirements on loading patterns. According to Liu and Hsiao (1997), the degree of stability between any two adjoining layers is mainly determined by the loading pattern. Besides the loading pattern, the stability of a pallet unit load also depends on the physical characteristics of the product, the mode of transportation and other factors. If the loading patterns are arranged properly, the unit load will be more stable. And Carpenter and Dowsland (1985) point out that the suitability of the pallet-loading patterns should be determined with respect to a variety of criteria affecting stability and clamp ability. Using compacting, centering blocks and distributing gap procedures, Bischoff extended the Bischoff and Dowsland (1982) algorithm to generate more stable layouts under the same criteria as those proposed by Carpenter and Dowsland. From the viewpoint of loading and handling the unit load, besides the loading pattern, the arrangement of boxes must meet the weight, stability and balance constraints (Gehring and Bortfeldt, 1997.)

The pinwheel-loading pattern, as a sound, textured and easy-to-implement pattern, has a longer history of practical application (Bolz and Hagernan, 1958; Tompkins et al., 2003). Its practical advantage lies in the following three aspects:

2.3.1. Stability

The stability of a pallet unit load is a prime requirement (Bischoff, 1991); proper loading stability will increase shipping stability, help to minimize transportation damage and reduce the amount of packing material needed (Liu and Hsiao, 1997). Because the pinwheel has a rectangular contour and close-loop H-box- and V-box-interlaced texture with an $R(180)$ symmetry rather than a reflection symmetry, we can apply a reflection in the X- or the Y-axis for the second layer, then all alternative layers are different with the box in one orientation interlocked by another orientated counterpart of alternative layers. Therefore, a quite stable alternative layer stacking is made through the odd and even layers' interlocking. This accordingly contributes to the stability of the whole pallet unit load.

2.3.2. Dynamic stability and balance

The stability of pallet load is more important when in a dynamic moving vehicle or handling situation. Both the pinwheel and the one-block pattern have a rigid rectangular contour, but the pinwheel will be more stable when in a dynamic vibrating situation because it has no "perfect column" structure like the one-block pattern. On the other hand, the pinwheel is a balanced pattern with a weight distribution better than two, three and even more block patterns. Its rotational symmetry means that the center of gravity is much closer to the contour's centroid, which can be easily derived from the examples in Fig. 1.

2.3.3. Contour and multi-face

In a pinwheel, the holes are defined less than a box by area; likewise, the base contact criterion is also satisfied. Furthermore, given a small enough hole (as in Fig. 1a, e, f, h), the pinwheel pattern would have a rigid rectangular contour; in both the X and the Y direction, the contour possesses a pair of perfectly flat opposite faces. Such pinwheel patterns are anti-slide and compact layouts with unused space inside but these holes cannot be moved out to contour edges through a simple "pushing" procedure. This is advantageous to a practical pallet-loading or clumping situation, where either palletizers, clump trucks or manual-loading methods are used to construct rigid and compact-loading layers or to handle for storage, stacking, loading/unloading, shrink-wrapping, etc. In a pinwheel pattern, the different outward faces of the outer boxes can be seen on each side of the pinwheel contour; thus, the label and other information on the box can be read or scanned more easily than homogeneous faces.

In all, the pinwheel pattern has practical advantages over the one-block pattern and other jag-contoured block patterns. It trades a small reduction in the number of boxes fitted for a more stable layout to implementation.

3. Pinwheel pattern by category

Because the pinwheel pattern has different holes (loops) arrangement, leaves formation and orientation, we classify pinwheel patterns into three categories: a simple pinwheel, a block pinwheel and a nested pinwheel, as shown in Fig. 1. These patterns become increasingly complex from a simple, block to nested pinwheel; while the first type has a single hole/loop and simple leaves of a single box, the block pinwheel has a single hole and at least a pair of leaves with blocks rather than single boxes, and the nested pinwheel has many nested loops/holes with the holes lying in a line or forming a hole-lattice.

The pinwheel is a texture with single or multiple loops/holes; hence, two parameters i and j are used to describe the hole layout, and the pinwheel loop consists of an interlaced H- and V-leaf, which may be a box block or a single box. Then an H-leaf is defined as m rows n columns of H-boxes and a V-leaf is defined as p rows q columns of V-boxes. Thus, a generic notation is recommended for a pinwheel with a hole-lattice of i rows (here, rows are parallel to the secondary diagonal – i.e. from the northeast corner to the southwest corner – of the contour rectangle formed by these boxes), j columns and with an H-leaf of m rows n columns and a V-leaf of p rows q columns boxes:

$$[i,j-]m \times n - p \times q.$$

For example, Fig. 1a shows a 1,1–1 × 1–1 × 1 simple pinwheel. If there is just one single hole, or to say no nested loop, the item in the square bracket can be ignored; Fig. 1c shows a 2 × 3–2 × 2 block pinwheel, and Fig. 1d and e shows a 4 × 2–1 × 3 and 4 × 1–1 × 6 block pinwheel, respectively. However, the nested pinwheel should retain i and j parameters, for example, Fig. 1f shows a 4,1-1 × 1-1 × 1, Fig. 1g shows a 1,2–3 × 1–1 × 4 and Fig. 1h shows a 2,3–1 × 1–1 × 2 nested pinwheel.

Also, by our notation, the hole-lattice is in i rows and j columns along the secondary diagonal, and there are i H-leaves and j V-leaves on the horizontal contour edge, with a length of $ina+jqb$. Likewise, the contour width is $imb+jpa$. Thus, the total boxes number N is:

$$N = (i+j)(imn + jpq). \tag{4}$$

Especially, for an $m \times n–p \times q$ block pinwheel, the total box number is $2(mn+pq)$ and a simple pinwheel with just four boxes.

3.1. Simple pinwheel

In a simple pinwheel, four boxes as leaves are arranged successively in a close-loop style, leaving a central hole. Both the contour of the pinwheel and the only central hole are squares. Simple pinwheels are 90° rotationally symmetric by the centroid; hence, denoted as $R(90)$. Because of the rotational symmetry, without loss of generality, we define the box on the left lower corner as the first leaf and it lies horizontally; hence, 1H. The second box lies vertically on the right lower corner (2V), the third horizontally on the right upper corner (3H) and the fourth finally on the left upper corner (4V), as shown in Fig. 1a in an anticlockwise sequence. The simple pinwheel pattern is a first-order non-guillotine pattern. A special case is when the area of the hole is zero. In fact, it is a guillotine pattern with square boxes.

3.2. Block pinwheel

Block pinwheels (Fig. 1c–e) differ from simple pinwheels in that at least a pair of leaves in a block pinwheel is no longer single boxes, but blocks with guillotine patterns of identical and uni-orientated boxes. A block pinwheel has the same anticlockwise leaf sequence as a simple pinwheel. The pattern contour of the block pinwheel is a rectangle. A rectangular contour is better and more practical for rectangular pallets, namely, $1200 \times 1000\,\text{mm}^2$ International (approximately, including the $48 \times 40\,\text{in.}^2$ USA common pallet) and $1200 \times 800\,\text{mm}^2$ Euro types. In a rectangle-shaped block pinwheel pattern, the pair of horizontal leaves is the same, and so is the vertical pair. Hence, the block pinwheel is 180° rotationally symmetric by the centroid, denoted as $R(180)$. The block pinwheel is also a first-order non-guillotine pattern and a natural outcome of many block-based algorithms (Nelißen, 1993). There are special cases when the area of the hole is zero and it is still a guillotine pattern. This is only possible when the box ratio is an integer.

As stated above, the values of i and j for the block pinwheel are all 1s, and so a shorter version of our notation may apply, that is $m \times n–p \times q$. If $m = q$ and $n = p$, it is a square-shaped pinwheel, which is a better fill to the JIS (1100×1100 or $1140 \times 1140\,\text{mm}^2$) square pallet. A simple pinwheel is just a special $1 \times 1–1 \times 1$ instance of the block pinwheel.

3.3. Nested pinwheel

By our classification, for a nested pinwheel pattern, one or more leaves of the pinwheel are further nested by other pinwheel(s), and more boxes/blocks are added to form a complete rectangle. Once they are nested, there will be more than one hole and each hole should be encompassed by a loop of leaves while a common leaf is shared by two adjacent loops.

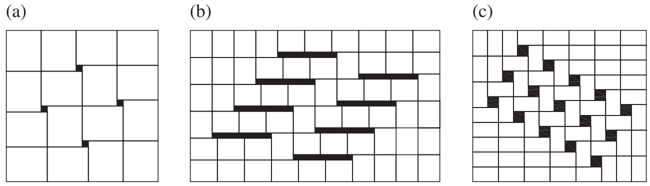

Fig. 2. Some examples of two-dimensional lattice nested pinwheels from left to right of the corresponding minimal size instances are (14,14,4,3), (102,63,11,9) and (81,71,12,7).

Based on the leaf conformation, nested pinwheels are shown in two subtypes: nested simple (S) and nested block (B). In a nested simple pinwheel (see Fig. 1f), each leaf only contains a single box as does the simple pinwheel. If the pinwheel leaf is not a block but further nested by a small pinwheel, this will be a fractal. However, it is beyond the scope of our discussion because our box is defined and beyond partition. Besides, based on the complexity of the hole lattice, the nested pinwheels are also shown in two subtypes: one-dimensional and two-dimensional. The one-dimensional lattice (1DL) is just a line of holes with either $i = 1$ or $j = 1$. But i and j cannot be both 1s; otherwise, it is a block pinwheel. The two-dimensional lattice (2DL) is an i-row and j-column array of holes with both $i, j > 1$. Thus, there are four subtypes of nested pinwheels, which are 1DL/S (Fig. 1f), 1DL/B (Fig. 1g), 2DL/S (Fig. 2a and c) and 2DL/B (Figs. 1h and 2b). 1DL/B nested block pinwheels differ from 1DL/S nested simple pinwheels in leaf conformation, just like the block pinwheels differ from simple pinwheels. The 1DL nested pinwheel has leaves (block or box) shared along the diagonal; such a shared leaf is also called a diagonal block (Exeler, 1988) or a hollow block (HB) in Martins and Dell's HB heuristic (2008), both Exeler's and HB heuristics can generate such styles. The number of H-boxes and V-boxes for 1DL conforms to the theorem 2 of Martins and Dell's (2008) paper. While 1DL nested pinwheel patterns are first-order non-guillotine patterns, 2DL nested pinwheel patterns are superior-order non-guillotine patterns that are more complex. To date, we have not yet found any heuristic, metaheuristic or exact algorithm that can generate such a 2DL pattern. No matter what the four subtypes are, a nested pinwheel has an $R(180)$ symmetry, and its total box number is applied by Equation (4). Especially, when $m = n = p = q = 1$, the box number in a 1DL/S or 2DL/S nested simple pinwheel is $(i+j)^2$.

Both the block pinwheel and the nested pinwheel are called split pinwheels in the *Early Material Handling Handbook* by Bolz and Hagernan (1958). At that time, there were no computers to generate the complex non-guillotine patterns; their patterns emerged from intuition and experience. This shows that the pinwheel pattern is common and of practical use (Tompkins et al., 2003). The beauty and accessibility of these geometric patterns makes the pinwheel pattern a preferable alternative pattern in pallet loading.

4. Geometry of pinwheel

The discussion of pinwheel geometry will be helpful to understand what the structural requirement for a pinwheel pattern is, and in what range the box ratio required by a specific pattern is.

4.1. Hole constraint

In a pinwheel, it was generally supposed that the central hole should be small enough not to allow any box to fill in, i.e. the area of the hole should be less than a box's area:

$$|pa - mb||na - qb| \leqslant ab.$$

By the pinwheel definition, if the first leaf (1H) is lower than the second leaf (2V), that is, a P1 style with $pa - mb > 0$ and $na - qb > 0$ or it is a P3 style with $pa - mb < 0$ and $na - qb < 0$. Thus, there is always:

$$(pa - mb)(na - qb) \leqslant ab. \qquad (5)$$

With $a = b$, β substitutes a in inequality (5), then

$$(p\beta - m)(n\beta - q) \leqslant \beta.$$

That is,

$$pn\beta^2 - (mn + pq + 1)\beta + mq \leqslant 0. \qquad (6)$$

Thus, we have

$$\Delta = b^2 - 4ac = (mn + pq + 1)^2 - 4mnpq = (mn - pq + 1)^2 + 4pq.$$

Because m, n, p and q are all integers no less than 1, $\Delta \geqslant 0$, then the β range is within

$$\frac{mn + pq + 1 - \sqrt{\Delta}}{2pn} \leqslant \beta \leqslant \frac{mn + pq + 1 + \sqrt{\Delta}}{2pn}. \qquad (7)$$

The value of root expression may be <1 but by definition $\beta \geqslant 1$. Thus, we can have the box shape ratio range for a given pinwheel pattern.

For a simple pinwheel, with $m = n = p = q = 1$, $\beta \leqslant 2.618$. This means that the box ratio cannot exceed 2.618. Otherwise, the area of the central hole is greater than the area of a box, and a box may (but not certainly can) be added on the hole. For example, if $\beta = 2.5$, and the largest possible box size is 715×285 on a 1200×1000 pallet. The area of the central hole is $(715–285)^2 = 184{,}900$, which is less than the area of a box (203,775).

4.1.1. Rigid condition

As shown in Figs. 1c, d, g and 2b, the holes are not rigid because a push inward can change the hole and thus create a ragged pinwheel contour. Here, a rigid condition should be derived. If the holes are square, as long as the inequality (5) is a strict inequality without equality, then the hole is rigid, as shown in Figs. 1a, f and 2a, c. Otherwise, four cases should be considered. The rectangular hole can have only two orientations, namely a H-hole (with a longer horizontal edge) and a V-hole, while there are two pinwheel pattern styles P1 and P3; thus, the four cases are P1-H, P1-V, P3-H and P3-V. For a P1-H hole, if $na - pb < a$, then the hole is small enough to slide, the pinwheel pattern is rigid. Thus, $\beta < p/(n - 1)$. Similarly, the rigid condition for the P1-V pinwheel is $\beta < (m+1)/p$; for the P3-H pinwheel it is $\beta > (q - 1)/n$; and for the P3-V pinwheel it is $\beta > (m - 1)/p$. Given the rigid condition and β range, rigid pinwheels can be determined.

4.2. Contour shape

As mentioned above, for an i, j–$m \times n$–$p \times q$ pinwheel, a total number of $(i+j)$ $(imn+jpq)$ boxes construct $i \times j$ pinwheel loops, form a rectangular pattern with a length of $(ina+jqb)$ and a width of $(imb+jpa)$. Let δ be the contour rectangle shape ratio, and with $a = b\beta$, then:

$$\delta = \frac{i \cdot n \cdot a + j \cdot q \cdot b}{j \cdot p \cdot a + i \cdot m \cdot b} = \frac{i \cdot n \cdot \beta + j \cdot q}{j \cdot p \cdot \beta + i \cdot m}. \tag{8}$$

By definition, δ should be no less than 1; otherwise, it can be turned 90° around to have a value >1.

On the other hand, given the pallet ratio α, if the pinwheel pattern fills the pallet deck board perfectly with $\delta = \alpha$, then Equation (8) can be transformed to compute the β value for such a pinwheel with a perfect pallet contour. Let β_0 be such a perfect value, then the equation is as follows:

$$\beta_0 = \frac{i \cdot m \cdot \alpha - j \cdot q}{i \cdot n - j \cdot p \cdot \alpha}. \tag{9}$$

Also, the δ value can be compared with the α value of any given $X \times Y$ pallet. In addition to the perfect fill, in most cases there will be some empty space along the X or the Y edge of the pallet. If $\delta > \alpha$, then

$$a = \frac{X \cdot \beta}{i \cdot n \cdot \beta + j \cdot q}; \qquad (10;)$$

otherwise,

$$a = \frac{Y \cdot \beta}{i \cdot n + j \cdot p \cdot b}. \tag{11}$$

These equations are the foundation for the calculations.

5. Calculation and discussion

Given the above pinwheel geometry, the β range for a reasonable pinwheel can be figured out for any pinwheel instance of i, j, m, n, p,and q combinations, and given a β value within that feasible range, an optimal box size on a given pallet dimension may be obtained. This is a reverse way to find the optimal pinwheel patterns by enumerating all pinwheel instances of i, j, m, n, p and q combinations.

5.1. Dataset

As mentioned above, the two datasets for PLP were Cover I with $1 \leqslant \alpha \leqslant 2$, $1 \leqslant \beta \leqslant 4$ and $4 \leqslant \gamma < 51$ and Cover II with $51 \leqslant \gamma < 101$ (Dowsland, 1984; Nelißen, 1995). These two sets of restrictions on pallet and box dimensions have always been used by other authors (e.g. Nelißen, 1993; Scheithauer and Terno, 1996; Morabito and Morales, 1998; Alvarez-Valdes et al., 2005; Birgin et al., 2005; Lins et al., 2003; Martins and Dell, 2008). However, the β range of 1–4 is derived from the box range with a common length $200\,\text{mm} \leqslant a \leqslant 600\,\text{mm}$, with width $150\,\text{mm} \leqslant b \leqslant 450\,\text{mm}$ (Dowsland, 1984). In

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practical situations, especially in factories, the box range varies and can exceed the limit significantly; thus, $\beta \leqslant 4$ limits many practical applications. Also, Alvarez-Valdes et al. (2005) point out that the definition of sets Cover I and Cover II is subject to some ambiguity. Sometimes, an instance satisfies the conditions defining the set but another equivalent instance does not. As one-block optimal patterns dominate PLP solutions (Martins and Dell, 2008), they are more likely to appear with a higher γ value. We think that the $\gamma < 51$ is slightly less and $\gamma < 101$ is slightly more, and an average $\gamma \leqslant 76$ of both Covers may be better. The relationship of the box ratio and the area ratio also needs to be re-examined. Hereby, we propose a unique dataset with the range of $1 \leqslant \beta \leqslant 10$ and $4 \leqslant \gamma \leqslant 76$.

Our calculations are all based on this new dataset. We have calculated all the 10,860 feasible instances of i, j, m, n, p, and q combinations in the above-proposed range. Also, for the real application of pinwheel patterns in pallet loading, we choose the ISO $1200 \times 1000\,\text{mm}^2$ pallet (P1210), the Euro $1200 \times 800\,\text{mm}^2$ pallet (P1208) and the Japan 1100 mm square pallet (P1111). Their α values are 1.2, 1.5 and 1.0, respectively. Because Martins and Dell (2007) have solved all the minimal size instances (MSI) of PLP, which is easy to obtain from their website, http://www.palletloading.org, we use these real PLP instances on three common pallets directly, and some examples and their MSIs are also listed in the tables below.

5.2. *"Perfect" pinwheel*

Combining Equations (7) and (9), pinwheels with a contour perfectly filled to a given pallet (hence called "perfect" pinwheels) can be computed by enumerating all possible instances of i, j, m, n, p and q combinations. After the perfect β_0 is determined, the PLP instance can be calculated based on Equations (10) and (11). Table 1 sums up these "perfect" pinwheels by category and by pallet, and Fig. 3 shows their distribution on the box number and box ratio matrix. However, some of

Table 1
"Perfect" pinwheel by category and by pallet

	Simple	Block	Nested		Total
			1DL	2DL	
Instances	1	7891	2668	300	10,860
P1111 perfect	1	26	122	16	165
$\beta_0 > 4$	0	0	38	0	38
$\beta_0 > 10$	0	0	2	0	2
Max β_0	2.718	4	11	3	11
P1208 perfect	0	80	78	14	172
$\beta_0 > 4$	0	44	14	3	61
$\beta_0 > 10$	0	23	2	0	25
Max β_0	0	29	12	8	29
P1210 perfect	0	62	113	14	189
$\beta_0 > 4$	0	34	44	0	78
$\beta_0 > 10$	0	19	2	0	21
Max β_0	0	25	10.86	4.125	25
Summary	1	168	313	44	509

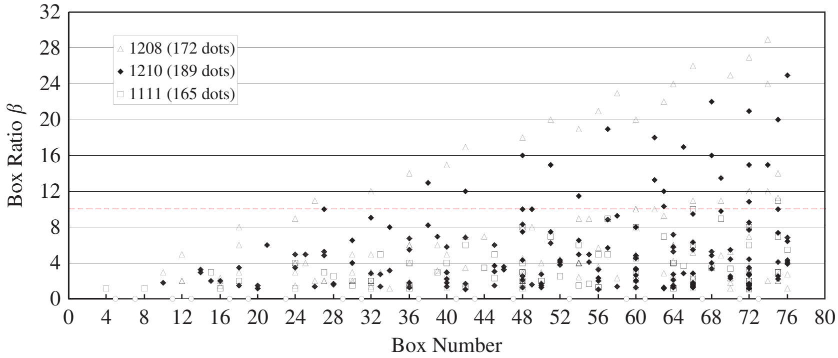

Fig. 3. Box number and β value distribution for pinwheels with a perfect pallet contour circle (o) on the horizontal axis indicates a prime number, which is impossible in a pinwheel.

these pinwheels may not be optimal because multiple holes may waste considerable inner area. Only if the rigid conditions are satisfied, are the perfect pinwheels rigid and anti-slide.

In Table 1, it can be seen that the "perfect" pinwheels appear most likely to be 1DL nested pinwheels and the next is block pinwheels; these first-order non-guillotine patterns count for 95.5% of the total "perfect" pinwheels. By pallet, the square pallet P1111 is more difficult to fill perfectly, especially with long boxes (Max β_0 is only 11). While Morabito et al.'s (2000) research shows that the P1210 pallet has the highest mean area utilization, our computation shows that it is also the most easy to fill perfectly. Among these 509 pinwheels, the largest $\beta_0 = 29$ is for 9×1–1×28, 74 boxes block pinwheel with a PLP instance of (1200, 800, 610 and 21), and it is optimal with a minimum size instance of (57, 38, 29 and 1). Using β_0 values, perfect pinwheels with $\beta_0 > 4$ account for about one-third, and pinwheels with $\beta_0 > 10$ account for nearly one-tenth of all pinwheels. This indicates that the percentage of the box ratio >4 is substantial. Figure 3 further illustrates the distribution of box ratios and shows that $>90\%$ of the pinwheels lie within the $\beta \leqslant 10$ area (below the dashed line), while the $\beta < 4$ area is the densest area.

5.3. β *range*

In most cases, the pinwheel contour will not fill the pallet perfectly even though the pinwheel pattern can be optimal to the PLP. Usually, a range is set to the contour shape ratio δ, that is, $\delta \in [(1-\lambda)\alpha, (1+\lambda)\alpha]$, where λ may be 5–10%. The data for some pinwheel examples listed in Table 2 are computed based on the above $\pm\lambda$ relaxation. Again, the pinwheel pattern may not be optimal, and the γ bound, and Martins and Dell's results are listed as a comparison. Here, the β range is computed by Equation (7) along with $\beta \geqslant 1$ by definition. In Table 2, there are 35 pinwheel examples for loading on the common pallets, while only 14 pinwheels are optimal. From the "Difference" column of Table 2 we can see that the difference of the pinwheel's box number to optimality is small. It seems that when the β value is larger, the pinwheel pattern tends to be optimal and usually with a block pinwheel pattern; when the box number is larger, the pinwheel pattern tends to non-optimal. Please note that for some instances, they may have multiple optimal patterns. For example, in line numbers 6, 13 and 21, optimal solutions are pinwheel patterns,

Table 2
Examples of pinwheel on pallets

No.	i	j	m	n	p	q	N	β range	Example β	X	Y	a	b	δ	γ	Optimal	Optimal solution by Martins and Dell's (2008) method[a] MSI	Boxes	Difference	Is PW	Pattern type
1	1	1	1	1	1	1	4[b]	1.000–2.618	2.510	1200	1000	715	285	1.0	5	No	(8,7,5,2)	5	1	No	2-Block
2	1	1	2	1	1	2	8	2.000–4.000	1.200	1100	1100	412	243	1.0	8	No	(8,8,3,2)	10	2	No	2-Block
3	2	1	1	1	1	1	9	1.000–2.618	1.750	1200	1000	466	266	1.200	10	Yes	(8,7,3,2)	9	0	Yes	Nested pinwheel
4	2	2	1	1	1	1	16[c]	1.000–2.618	1.272	1100	1100	308	242	1.000	16	Yes	(14,14,4,3)	16	0	Yes	Block pinwheel
5	2	1	2	1	1	2	18	2.000–4.000	3.500	1200	1000	466	133	1.2	19	No	(18,15,7,2)	19	1	No	3-Block
6	1	1	2	3	2	2	20[d]	1.000–1.333	1.07	1200	1000	245	230	1.2	21	Yes	(28,24,6,5)	20	0	No	1-Block
7	1	1	4	1	1	6	20[e]	3.000–8.000	6.53	1200	1000	620	95	1.19	20	Yes	(25,21,13,2)	20	0	Yes	Block pinwheel
8	1	2	1	1	1	3	21	3.000–4.302	3.429	1200	1000	436	127	1.2	23	Yes	(28,23,10,3)	21	0	Yes	Nested pinwheel
9	1	1	4	2	1	3	22[f]	1.268–4.732	1.45	1200	1000	266	183	1.083	24	No	(13,11,3,2)	23	1	No	3-Block
10	2	2	1	1	1	2	24	2.000–3.414	2.500	1200	1000	333	133	1.286	27	No	(18,15,5,2)	27	3	No	1-Block
11	4	1	1	1	1	1	25[g]	1.000–2.618	1.204	1200	1000	252	186	1.196	25	Yes	(33,27,7,5)	25	0	Yes	Nested pinwheel
12	1	1	3	1	1	10	26	10.000–11.359	10.679	1200	800	620	58	1.512	27	Yes	(41,27,21,2)	26	0	Yes	Block pinwheel
13	1	2	3	1	1	4	33[h]	2.000–6.000	2.770	1200	1000	305	110	1.261	35	Yes	(43,36,11,4)	33	0	No	2-Block
14	1	4	1	3	1	1	35	1.000–1.434	1.300	1200	1000	197	151	1.274	40	No	(31,26,5,4)	40	5	Yes	Block pinwheel
15	2	1	4	1	1	4	36	4.000–6.561	5.333	1100	1100	400	75	1.100	40	No	(14,14,5,1)	39	3	No	4-Block
16	1	2	3	1	1	5	39	2.209–6.791	2.75	1200	800	258	93	1.5	40	No	(51,34,11,4)	39	0	Yes	Nested pinwheel
17	1	1	3	2	2	7	40	3.500–3.905	3.7	1200	1000	308	83	1.384	46	No	(58,48,15,4)	45	5	No	2-Block
18	2	3	1	1	1	2	40[i]	2.000–3.414	2.254	1200	1000	257	114	1.199	40	Yes	(42,35,9,4)	40	0	Yes	Nested pinwheel
19	1	5	1	3	1	1	48	1.000–1.434	1.200	1200	1000	167	139	1.229	51	No	(43,35,6,5)	50	2	No	3-Block
20	6	1	1	1	1	1	49	1.000–2.618	1.809	1200	800	183	102	1.518	51	No	(46,31,7,4)	50	1	No	3-Block
21	1	2	3	1	1	7	51	7.000–8.541	7.771	1200	1000	419	53	1.174	54	Yes	(66,55,23,3)	51	0	No	Guillotine
22	2	1	4	2	1	1	51	1.000–4.562	4.4	1200	800	283	64	1.5	53	Yes	(55,37,13,3)	51	0	Yes	Nested pinwheel
23	3	1	3	1	1	5	56	5.000–6.791	5.896	1200	800	311	52	1.523	59	No	(23,15,6,1)	57	1	No	4-Block
24	2	4	1	3	1	1	60[j]	1.000–1.434	1.217	1200	800	130	105	1.646	70	No	(56,37,6,5)	68	8	No	3-Block
25	2	5	1	2	1	1	63	1.000–1.707	1.31	1200	1000	152	116	1.198	68	No	(31,26,4,3)	67	4	No	4-Block
26	5	3	1	1	1	1	64[k]	1.000–2.618	1.735	1200	1000	170	98	1.144	73	No	(49,41,7,4)	71	7	No	2-Block
27	6	2	1	1	1	1	64	1.000–2.618	1.446	1200	1000	162	112	1.201	66	No	(52,44,7,5)	65	1	No	4-Block
28	5	3	1	1	1	1	64	1.000–2.618	2.149	1200	1000	187	87	1.201	73	No	(83,69,13,6)	71	7	No	4-Block

Table 2. (Contd.)

No.	i	j	m	n	p	q	N	β range	Example								Optimal solution by Martins and Dell's (2008) method[a]				
									β	X	Y	a	b	δ	γ	Optimal	MSI	Boxes	Difference	Is PW	Pattern type
29	3	2	2	2	1	1	70	2.000–2.618	2.309	1200	800	173	75	1.493	74	No	(48,32,7,3)	73	3	No	Guillotine
30	3	3	2	1	1	2	72	1.000–4.000	1.202	1100	1100	137	114	1	77	No	(48,48,6,5)	76	4	No	5-Block
31	1	2	7	3	1	2	75	7.000–7.367	7.183	1200	1000	336	46	1.196	77	No	(78,65,22,3)	76	1	No	First non-guillotine
32	1	2	7	1	1	9	75	5.458–11.541	11.27	1100	1100	417	37	0.991	78	Yes	(29,29,11,1)	75	0	Yes	Block pinwheel
33	1	2	3	3	2	4	75	1.000–2.000	1.267	1200	800	121	103	1.5	77	Yes	(69,46,7,6)	75	0	Yes	Nested pinwheel
34	1	3	4	4	1	1	76	4.000–4.266	4.133	1200	1000	252	60	1.191	79	No	(80,67,17,4)	78	2	No	First non-guillotine
35	3	1	4	1	1	7	76	7.000–8.828	7.914	1200	800	308	38	1.544	82	No	(127,84,33,4)	79	3	No	First non-guillotine

[a]These results are computed from the Martins website http://www.palletloading.org.,
[b]As shown in Fig. 1a.
[c]As shown in Fig. 2a.
[d]As shown in Fig. 1c.
[e]As shown in Fig. 1e.
[f]As shown in Fig. 1d.
[g]As shown in Fig. 1f.
[h]As shown in Fig. 1g.
[i]As shown in Fig. 1h.
[j]As shown in Fig. 2b.
[k]As shown in Fig. 2c.
MSI, Minimal Size Instance; UB, Computed Upper Bound; Is PW, Is pinwheel.

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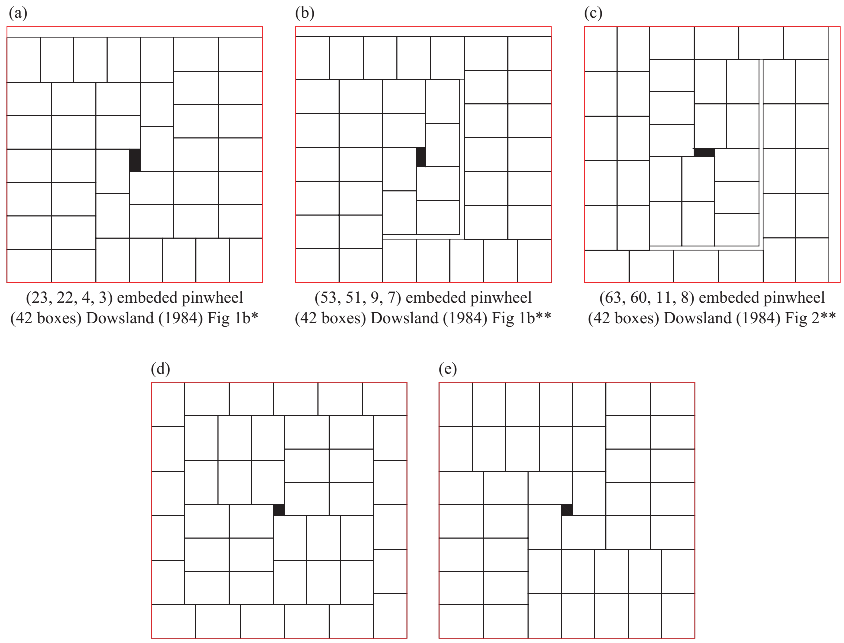
block patterns or guillotine patterns. Note, in the third row of the table, a nine-box 2,1–1 × 1–1 × 1 pinwheel has the smallest odd number of boxes among any of the pinwheel patterns. Also, there are 13 "perfect" pinwheels in Table 2.

In all, our two types of results verify our suggested box ratio limit to 10; also, this higher box ratio is not rare in real-life box dimensions. Because there are hard examples that many algorithms are working for, how about the new β limit, are there any new hard classes rising? This question should be answered in the future.

So far, the pinwheel pattern can be constructed by any number of boxes, except for those with prime numbers. For each i, j, m, n, p, q combination, there is a β range for the hole smaller than a box, and a rigid condition requires further β constraints. It may be an optimal pattern only when its contour ratio is close to the target pallet ratio. Our pinwheel pattern is not to pursue optimal box numbers but a near-optimum solution with a satisfactory number and pattern shape.

5.4. Slack of hole constraint

However, with our hole constraint, a special type of nested pinwheel – a pinwheel embedded completely within another pinwheel – cannot be formulated. Such "embedded pinwheel" examples can be found in Dowsland (1984, figs. 1b and 2) and Nelißen's (1993, fig. 19) examples of (23,23,4,3) or (40,40,7,5) instances by a complex block heuristic. These examples are shown in Fig. 4 below. Thus, we supplement our notation system by naming the "embedded pinwheel" as $m_1 \times n_1$–



(a) (23, 22, 4, 3) embeded pinwheel (42 boxes) Dowsland (1984) Fig 1b*

(b) (53, 51, 9, 7) embeded pinwheel (42 boxes) Dowsland (1984) Fig 1b**

(c) (63, 60, 11, 8) embeded pinwheel (42 boxes) Dowsland (1984) Fig 2**

(d) (23, 23, 4, 3) embeded pinwheel (44 boxes) Nelissen (1993) Fig 19a

(e) (23, 23, 4, 3) embeded pinwheel (44 boxes) Nelissen (1993) Fig 19b

Fig. 4. Embedded pinwheel examples. *New finding from the original pattern structure; **revised based on the original figure.

$p_1 \times q_1 \rightarrow m_2 \times m_2$–$p_2 \times q_2$, where the arrow indicates the embedding direction: the former small pinwheel is embedded into a later larger one. Then the Dowsland examples are denoted as 3×1–$2 \times 2 \rightarrow 1 \times 4$–$5 \times 2$ and 2×1–$2 \times 1 \rightarrow 1 \times 5$–$6 \times 2$, respectively, Nelißen's as 3×2–$2 \times 3 \rightarrow 1 \times 5$–$5 \times 1$ and 1×1–$1 \times 1 \rightarrow 5 \times 2$–$2 \times 5$. Only in a few cases, can the hole space of the outer loop be fully filled by the inner pinwheel, as shown in Fig. 4a, d and e; if it cannot be fully filled, the pattern contour is indeed jagged by pushing inward.

6. Asymmetric pinwheels

The pinwheel, as its name suggests should be symmetric to rotate. However, if the concept is extended to an asymmetric pinwheel, some interesting results for the loading patterns may be obtained. If a pinwheel does not have any identical leaves or full number of leaves (including sharing of leaves by adjacent loops) in a loop, then we call it an asymmetric pinwheel. It can be categorized as an asymmetric block, asymmetric nested or an incomplete pinwheel, as shown in Fig. 5. These

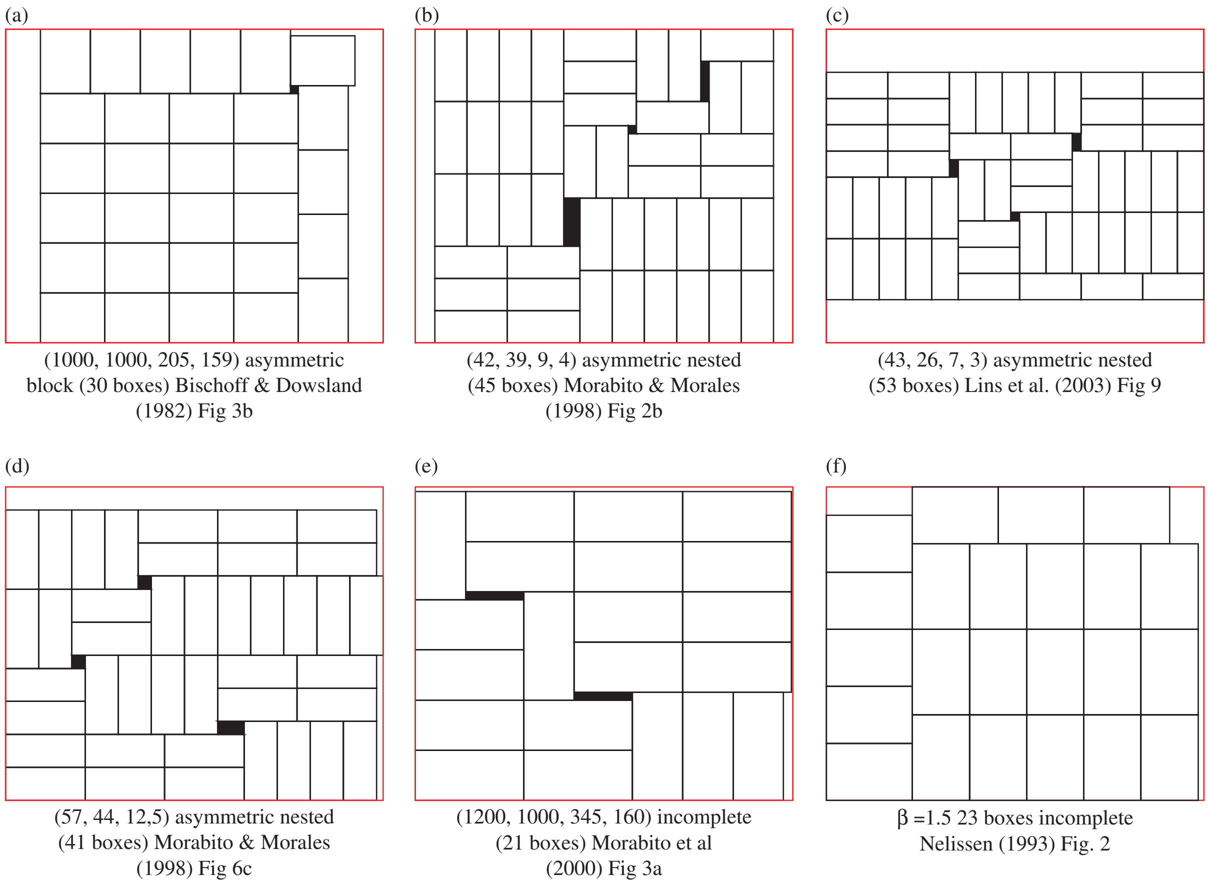


Fig. 5. Examples of asymmetric patterns (pushed and compacted).

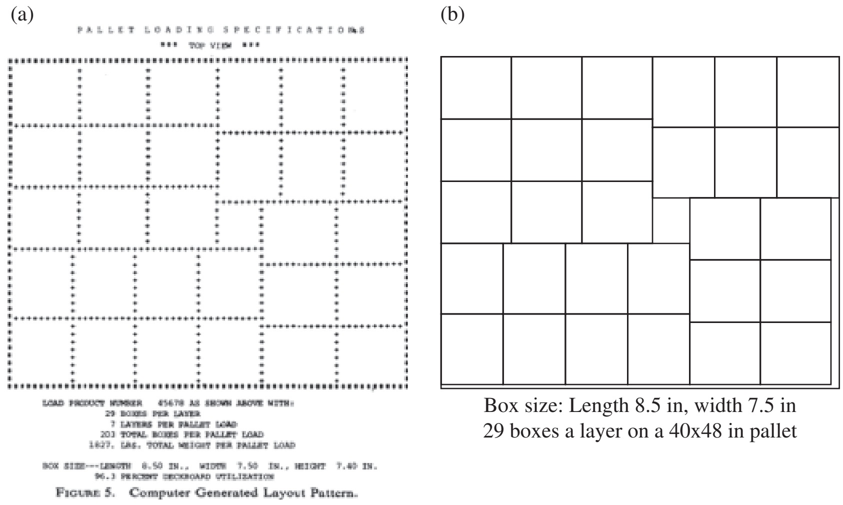


Fig. 6. Steudel's example pattern (a) and the real-scale drawing (b).

figures are from the literature, but the patterns are condensed to compact structures, which reveal some jagged edges.

As early as in Steudel's (1979) paper, his projected procedure of the two-phase solution produces an asymmetric pinwheel with four different leaves and jagged contour edges. Also, there was a misleading layout pattern shown in Fig. 6a. This is an asymmetric pinwheel, but it appears as though the pattern has a perfect rectangular contour. Indeed, the technical in-scale drawing of the pattern shown in Fig. 6b reveals the eye illusion of the early computer line text printed figure – it has a clear jagged edge on the right! However, Steudel's figure also gives us a hint: an asymmetric pinwheel without a rectangular contour is also acceptable in practice, as long as the jagged teeth are within certain limits.

An asymmetric block pattern can be described using a notation as $m_1 \times n_1$–$p_1 \times q_1$–$m_2 \times n_2$–$p_2 \times q_2$, and it starts from the lower left leaf to the upper left, counter-clockwise. The length of the opposite edges can be lightly different; here, we suggest a 3% allowance in a contour edge jag. For example, the jag of Steudel's asymmetric block is 2.08%. The contour-jagged asymmetric block pinwheel pattern is the outcome of the four block algorithms (Steudel, 1979; Smith and De Cani, 1980) and it is a first-order non-guillotine pattern.

Asymmetric nested pinwheel patterns may outperform other asymmetric ones because they may have a completely rectangular contour, as shown in Fig. 5b. For any nested pinwheel only if both i and j are >1 (with a hole lattice) will it be a superior-order non-guillotine pattern. However, the symmetric pinwheel notation cannot apply to this asymmetric one, but this similar pattern type also applies to asymmetric nested pinwheels, as in Fig. 5b, it is a first-order non-guillotine pattern for it only has a line of holes, while Fig. 5c is a superior-order non-guillotine pattern as it has a hole lattice. These complete rectangular-contoured patterns can be generated by the L-approach (Lins et al., 2003) and HONG (Martins and Dell, 2008). It appears that the L-approach is more likely to generate rectangular contours.

On the other hand, an incomplete pinwheel is a guillotine pattern, which indeed is a combination of pinwheels and/or some other pinwheel-loop-free blocks. At least there is a guillotine cut traversing to one pair of the contour edges (see Fig. 5e and f for example). Two-block, three-block (Nelißen, 1993), G4 (Scheithauer and Terno, 1996) and Morabito and Morales (1998) recursive heuristics generate incomplete pinwheel patterns.

As calculated in the symmetric pinwheel above, the box number for any symmetric pinwheel pattern cannot be a prime number. In the 4–76 range, there are 19 prime numbers, which accounts for a quarter of the range. Namely, they are 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71 and 73 (as the circles shown on the horizontal axis of Fig. 3). These numbers can only be rounded to even numbers for symmetric pinwheel patterns; nevertheless, these numbers can be loaded into asymmetric pinwheel patterns without any rounding. The asymmetric pinwheel patterns can reach optimality with these prime numbers of boxes. Examples are (14, 10, 3, 2) instances with 23 boxes optimal (Dowsland, 1984), (57, 44, 12, 5) with 41 boxes (Scheithauer and Terno, 1996) and (124, 81, 21, 10) with 47 boxes (Nelißen, 1993). Because all difficult equivalent classes of PLP instances with a ratio <101 are solved by special heuristics (Martins and Dell, 2008), some attention to these prime number instances and their layout pattern may be of interest and may be worthwhile.

7. Concluding remarks

In this paper, we presented the pinwheel pattern as an alternative to the PLP. The definitions, elements and practical advantages, generating algorithms of the pinwheel patterns, were discussed. A uniform notation is proposed along with the identification of three types of symmetric pinwheels: simple, block and nested. With this notation, any pinwheel pattern can be defined and generated. The geometry of pinwheels is also studied and it provides the basis for our computation; it shows that each pinwheel pattern has a range of box ratio, and a pinwheel pattern may achieve optimality, especially for a larger box ratio. The ranges for all pinwheel instances with an area ratio of no more than 76 boxes are calculated and especially those pinwheels that can fill the pallet perfectly are calculated and investigated. A pinwheel pattern can be found for all non-prime number of boxes. Furthermore, we identified the practical advantages of a pinwheel pattern as: a uniform notation, an orderly textured structure, an $R(180)$ rotational symmetry and a sound contour. These advantages enable pinwheel patterns to be a good practical solution. As discussed in the last section, the asymmetric pinwheel extends the application of the pinwheel pattern: the box number can be a prime number for an asymmetric block pinwheel, a nested pinwheel and an incomplete pinwheel; and they can be easily found by many existing algorithms.

The characteristics of the pinwheel pattern are finally summarized in Table 3. With this systematic analysis, we conjecture that the pinwheel pattern is an advantageous alternative solution to the MPLP. It can be used widely by manufacturers and distributors and carriers in factories, trucks, vehicles, warehouses and distribution centers, and it is also helpful in the packaging design.

Nevertheless, a drawback of the pinwheel is that it cannot guarantee optimality in area utilization. Also, the box ratio is too short ranged to meet the requirement of all instances. Many existing algorithms can generate a pinwheel pattern, but no algorithm is found to be able to generate it with certainty. In our study, some issues are also raised, such as the old ranges of the

Table 3
Pinwheel-loading pattern in summary

Type	Notation	Box number	Symmetry	Contour[a]	Pattern[b]	Example
Simple	$1 \times 1–1 \times 1$	4	$R(90)$	S	First	Fig. 1a
Block	$m \times n–p \times q$	$2(mn+pq)$	$R(180)$	R	First	Fig. 1c, d, e
Nested						
1DL/S	$i,1–1 \times 1–1 \times 1$ or $1, j–1 \times 1–1 \times 1$	$(i+1)^2$ or $(j+1)^2$	$R(180)$	R	First	Fig. 1f
1DL/B	$i,j–m \times n–p \times q$ ($i = 1, j > 1$ or $j = 1, i > 1$)	$(i+j)\ (imn+jpq)$	$R(180)$	R	First	Fig. 1g
2DL/S	$i,j–1 \times 1–1 \times 1$	$(i+j)^2$	$R(180)$	R	Superior	Fig. 2a, c
2DL/B	$i,j–m \times n–p \times q$	$(i+j)\ (imn+jpq)$	$R(180)$	R	Superior	Fig. 1h, Fig. 2b
Embedded	$m_2 \times n_2–p_2 \times q_2 \rightarrow m_2 \times n_2–p_2 \times q_2$	$2(m_1n_1+p_1q_1) +2(m_2n_2+p_2q_2)$	$R(180)$	Rmj	First	Fig. 4
Asymmetric						
Block	$m_1 \times n_1–p_1 \times q_1–m_2 \times n_2–p_2 \times q_2$	$m_1n_1+p_1q_1+ m_2n_2+p_2q_2$	No	JR	First	Fig. 5a
Nested	NA	NA	No	Rmj	First or Superior	Fig. 5b, c, d
Incomplete	NA	NA	No	JR	Guillotine	Fig. 5e, f

[a]Contour type: S, square; R, rectangle; Rmj, rectangle (may jagged); JR, jagged rectangle.
[b]Pattern type: First, first-order non-guillotine; superior, superior-order non-guillotine.

box ratio (from 1 to 4) need to be re-examined: what is the best pattern for those situations with prime numbers of boxes? With these questions answered, we may set the epilogue of the MPLP.

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References

Alvarez-Valdes, R., Parreñob, F., Tamarita, J.M., 2005. A branch-and-cut algorithm for the pallet loading problem. *Computers & Operations Research* 32, 3007–3029.

Amaral, A., Wright, M., 2001. Experiments with a strategic oscillation algorithm for the pallet loading problem. *International Journal of Production Research* 39, 2341–2351.

Arenales, M., Morabito, R., 1995. An and/or-graph approach to the solution of two-dimensional non-guillotine cutting problems. *European Journal of Operational Research* 84, 599–617.

Beasley, J.E., 1985. An exact two-dimensional non-guillotine cutting tree search procedure. *Operations Research* 33, 49–64.

Bischoff, E.E., 1991. Stability aspects of pallet loading. *OR Spectrum* 13, 189–197.

Bischoff, E.E., Dowsland, W.B., 1982. An application of the micro to product design and distribution. *Journal of the Operational Research Society* 33, 271–281.

Bolz, H.A., Hagernan, G.E. (eds.), 1958. *Material Handling Handbook*. Ronald Press, New York.

Carpenter, H., Dowsland, W.B., 1985. Practical considerations of the pallet-loading problem. *Journal of the Operational Research Society* 36, 489–497.

Dowsland, K.A., 1984. The three-dimensional pallet chart: An analysis of the factors affecting the set of feasible layouts for a class of two-dimensional packing problems. *Journal of the Operational Research Society* 35, 895–905.

Dowsland, K.A., 1985. Determining an upper bound for a class of rectangular packing problems. *Computers and Operations Research* 12, 201–205.

Dowsland, K.A., 1987a. An exact algorithm for the pallet loading problem. *European Journal of Operational Research* 31, 78–84.

Dowsland, K.A., 1987b. A combined database and algorithmic approach to the pallet loading problem. *Journal of the Operational Research Society* 38, 341–345.

Dowsland, K.A., Dowsland, W.B., 1992. Packing Problems. *European Journal of Operational Research* 56, 2–14.

Dyckhoff, H., 1990. A typology of cutting and packing problems. *European Journal of Operational Research* 44, 145–159.

Exeler, H., 1988. *Das homogene Packproblem in der betriebswirtschaftlichen Logistik*. Physica Verlag, Heidelberg.

Gehring, H., Bortfeldt, A., 1997. A Genetic Algorithm for Solving the Container Loading Problem. *International Transactions in Operations Research* 4, 401–418.

Herbert, E.A., Dowsland, K.A., 1996. A family of genetic algorithms for the pallet loading problem. *Annals of Operations Research* 63, 415–436.

Letchford, A.N., Amaral, A., 2001. Analysis of upper bounds for the Pallet Loading Problem. *European Journal of Operational Research* 132, 582–593.

Lins, L., Lins, S., Morabito, R., 2003. An L-approach for packing (l, w)-rectangles into rectangular and L-shaped pieces. *Journal of the Operational Research Society* 54, 777–89.

Liu, F.F., Hsiao, C., 1997. A Three-Dimensional Pallet Loading Method for Single-Size Boxes. *Journal of the Operational Research Society* 48, 726–735.

Martins, G.H.A., Dell, R.F., 2007. The minimum size instance of a pallet loading problem equivalence class. *European Journal of Operational Research* 179, 17–26.

Martins, G.H.A., Dell, R.F., 2008. Solving the pallet loading problem. *European Journal of Operational Research* 184, 429–440.

Morabito, R., Farago, R., 1998. A tight Lagrangean relaxation bound for the manufacturer's pallet loading problem. *Studia Informatica Universalis* 2, 57–76.

Morabito, R., Morales, S.R., 1998. A simple and effective recursive procedure for the manufacturer's pallet loading problem. *Journal of the Operational Research Society* 49, 819–828.

Morabito, R., Morales, S.R., Widmer, J.A., 2000. Loading optimization of palletized products on trucks. *Transportation Research Part E* 36, 285–296.

Nelißen, J., 1993. New approaches to the pallet loading problem, Working paper, RWTH Aachen. Available at ftp://ftp.informatik.rwth-aachen.de/pub/reports/others/pallet.ps.Z.

Nelißen, J., 1995. How to use structural constraints to compute an upper bound for the pallet loading problem. *European Journal of Operational Research* 84, 662–680.

Pureza, V., Morabito, R., 2006. Some experiments with a simple tabu search algorithm for the manufacturer's pallet loading problem. *Computers and Operational Research* 33, 804–819.

Scheithauer, G., Terno, J., 1996. The G4-heuristic for the pallet loading problem. *Journal of the Operational Research Society* 47, 4, 511–522.

Smith, A., De Cani, P., 1980. An algorithm to optimize the layout of boxes in pallets. *Journal of the Operational Research Society* 31, 573–578.

Steudel, H.J., 1979. Generating pallet loading patterns: A special case of the two-dimensional cutting stock problem. *Management Science* 25, 997–1004.

Tompkins, J.A., White, J.A., Bozer, Y.A., Tanchoco, J.M.A., 2003. *Facilities Planning*, (third edn), John Wiley & Sons, Toronto , Canada.

Wächer, G., Haußner, H., Schumann, H., 2007. An improved typology of cutting and packing problems. *European Journal of Operational Research* 183, 1109–1130.

Winston, W.L., 1994. *Operations Research: Applications and Algorithm*, (3rd edn), Wadsworth Inc., Belmont, CA.

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